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Finite Element Methods Applied to Nearly One-Way Wave Propagation*

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A numerical method is developed for approximately solving stationary wave propagation problems for which most of the energy propagates in an angle band of less than 90° about a fixed direction. This property often occurs in realistic physical models. The numerical method is based on factoring out the principal propagation direction in the solution and then applying the finite element method. The method is applied to wave propagation in a duct or waveguide and to the exterior scattering problem. It is shown both theoretically and by means of numerical examples that this method requires considerably less grid points and computational cost than standard discretization methods.

1. INTRODUCTION

The numerical solution of Helmholtz type equations is important in many physical problems describing the propagation and scattering of time-harmonic waves in general geometries. A common feature in many of these problems is that most of the energy propagating in a given coordinate direction is contained in a narrow angle band (not necessarily about the horizontal) and very little backscattering is present. This occurs, e.g., in connection with radio wave diffraction problems [9], laser beam propagation [13], electromagnetic wave propagation in waveguides [19], plasma physics [5], seismic wave propagation [7], and underwater acoustic propagation [18].

To clarify the concept of nearly one-way wave propagation, consider the following model of long range acoustic wave propagation in the ocean. Using cylindrical coordinates, the acoustic pressure $u(r, z, \Phi)$ corresponding to a point source satisfies the Helmholtz equation

$$\begin{aligned}(\Delta + K^2 n^2(r, z, \Phi))\, u &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \Phi^2} + K^2 n^2 u \\ &= -\frac{4\pi}{r} p_s \delta(z - z_s)\,\delta(r)\end{aligned} \tag{1.1}$$

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with suitable boundary conditions, where z is depth, r is the horizontal range, Φ is the azimuthal angle, K is a reference wave number, and $n(r, z, \Phi)$ is the index of refraction. The source strength is p_s and it is located at $r=0$, $z=z_s$. In a great many ocean environments, it can be seen [18] that for long range the material inhomogeneities (e.g., $n(r, z, \Phi)$) are slowly varying and the acoustic energy propagating outside of a narrow angle band is attenuated by bottom interactions. Under these assumptions, it follows that for sufficiently large r we have $u(r, z, \Phi) = U(r, z, \Phi)\, H_0^1(Kr)$, where U is slowly varying in r and the Hankel function satisfies $H_0^1(Kr) \approx (2/\pi i K r)^{1/2}\, e^{iKr}$ for $Kr \gg 1$. Hence it follows that for $Kr \gg 1$, we have

$$U(r, z, \Phi) \approx (\pi i K r/2)^{1/2}\, e^{-iKr} u(r, z, \Phi) \tag{1.2}$$

and U satisfies

$$\frac{\partial^2 U}{\partial r^2} + 2iK\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \Phi^2} + K^2(n^2-1)\, U = 0. \tag{1.3}$$

The important point is that the r-derivatives of U are much smaller than those of u.

Much work has been done to solve one-way propagation problems numerically using parabolic equation type methods (see [5, 7, 9, 13, 18, 19]). In such cases, the elliptic boundary value problem is approximated by a parabolic initial value problem that may be efficiently solved numerically by a marching algorithm. For example, if it is assumed that $|\partial^2 U/\partial r^2| \ll |2iK(\partial U/\partial r)|$ (paraxial approximation) in (1.3), then we obtain the parabolic equation

$$2iK\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \Phi^2} + K^2(n^2-1)\, U = 0. \tag{1.4}$$

While such methods are often quite effective, it is not always clear how much accuracy is lost when making the paraxial approximation. For example, while Eq. (1.4) is well suited for describing the propagation of sound in the far field of the water column, this may not be the case near an interface of the water column and the sea floor (due in part to a discontinuity in n at this interface and the potentially much larger variations in n in the sea floor). Furthermore, this will in general not be the case in the near field. It is still expected, however, that the propagation will occur, for the most part, in an angle band of less than 90° about one principal direction.

An alternative approach is to discretize the original elliptic problem, (1.1), using a finite element or finite difference method. This approach does not depend on or utilize the fact that there is a distinguished principal propagation direction. For large frequencies or large computational domains, the resulting system of equations can be extremely large. An iterative method has been recently developed, [2], to efficiently solve these non-selfadjoint, indefinite systems of equations. However; it is clear that a method for reducing the number of equations would greatly help both in reducing computational cost and in enlarging the scope of the numerical method.

In this paper, we shall describe two finite element methods for approximately solving the full elliptic problem that allows for a dramatic reduction in the number of equations when the propagation is nearly uni-directional. The methods are based on factoring out the main propagation direction in the solution. Consider, for example, problem (1.1). Method I consists of expressing (1.1) as a variational problem, then replacing u by U (using (1.2)) in this variational formulation, and finally discretizing the new variational problem for U using a standard piecewise polynomial finite element trial space. (This method was analyzed by the author in [11] in connection with the exterior scattering problem. However, no numerical results were presented there.) Method II consists of transforming Eq. (1.1) into (1.3) (again using (1.2)) and then applying the finite element method using a standard piecewise polynomial trial space to approximate U. Since U is much smoother than u with respect to r, both methods result in a considerable reduction in the number of r-grid points and hence in the computational cost. This will be demonstrated both for propagation in a duct or waveguide and for the exterior scattering problem. Note that finite difference methods could also be developed that take advantage of the smoothness of U, however, we only consider finite element methods.

We close this section by briefly outlining the remainder of the paper. In Section 2, we make more precise the definition of nearly one-way wave propagation. Furthermore, we demonstrate this for both the exterior problem and propagation in a waveguide. We also consider various ways of choosing a suitable reference wave number. In Section 3 we describe and analyze Methods I and II in the context of propagation in a waveguide. We shall see that the choice of test space can have an important bearing on the effectiveness of both methods and that in general the best results are obtained by incorporating the main propagation direction in the standard piecewise polynomial test space. (For example, in connection with problem (1.1), the test space would consist of piecewise polynomials multiplied by e^{iKr}.) In this case it is shown that Methods I and II are equivalent. In Section 4 we present numerical results demonstrating the effectiveness of this method both for propagation in a waveguide and the exterior scattering problem. In Section 5 we summarize our results.

2. Nearly One-Way Wave Propagation

We consider the Helmholtz equation

$$(-\Delta - K^2 n(x))\, u(x) = f(x) \qquad \text{in } \Omega, \tag{2.1}$$

where K is real, Ω denotes an unbounded region in R^2 (or R^3), $n(x)$ denotes the index of refraction, and u satisfies some specified boundary condition on $\partial\Omega$ and an outgoing radiation condition at infinity. Suppose that $x = (\rho_1, \rho_2)$ denotes an arbitrary point in Ω expressed in a convenient coordinate system and Δ denotes the

Laplacian expressed in this coordinate system. We shall use standard notation for the Sobolev spaces $H^l(B)$ with l a non-negative integer and B a bounded subset of R^2. To be precise, set

$$\|v\|_{H^l(B)} \equiv \left(\sum_{j=0}^{l} |v|^2_{H^j(B)} \right)^{1/2},$$

with the seminorm, $|\,|_{H^j(B)}$, defined by

$$|v|_{H^j(B)} \equiv \left(\sum_{i=0}^{j} \int_B |D^i_{\rho_1} D^{j-i}_{\rho_2} v(x)|^2 \, dx \right)^{1/2}.$$

$H^l(B)$ consists of all complex-valued functions v defined on B such that $\|v\|_{H^l(B)} < \infty$. Thus $H^l(B)$ consists of square integrable functions with square integrable derivatives up to order l.

We define nearly one-way propagation with respect to the ρ_1 coordinate direction as follows. Assume that for each integer $m \geqslant 1$ and some real number $K' \neq 0$, we have

$$u(x) = e^{iK'\rho_1} U(x), \tag{2.2}$$

with

$$\int_B |D^j_{\rho_1} D^{m-j}_{\rho_2} U(x)|^2 \, dx = \delta \int_B |D^j_{\rho_1} D^{m-j}_{\rho_2} u(x)|^2 \, dx, \qquad j = 1,..., m, \tag{2.3}$$

where $\delta \ll 1$ is independent of K. In many cases, we shall see that $\delta \to 0$ as $K \to \infty$. We shall also see that δ is often a function of the set $B \subset \Omega$. It follows from (2.3) that U is a much smoother function of ρ_1 then u. Note that the expressions on both sides of (2.3) contain at least one derivative with respect to ρ_1. In many cases of practical interest, the ρ_2-derivatives of u will be small compared to its ρ_1-derivatives. Hence our definition could be strengthened by adding the following condition to (2.3):

$$\int_B |D^m_{\rho_2} U(x)|^2 \, dx \ll \int_B |D^m_{\rho_1} u(x)|^2 \, dx, \; m \geqslant 1. \tag{2.3'}$$

This is frequently the case when the paraxial approximation is employed. Note that the definition (2.2), (2.3) of nearly one-way propagation could be readily extended to higher dimensions.

We next illustrate the definition (2.2), (2.3) in the context of wave propagation in a duct or waveguide. For simplicity, we consider the following two-dimensional problem in Cartesian coordinates, $x = (x_1, x_2)$:

$$(-\Delta - K^2 n(x))\, u(x) = 0 \text{ in } \Omega. \qquad \frac{\partial u(0, x_2)}{\partial x_1} = g(x_2),$$
$$\frac{\partial u(x_1, 0)}{\partial x_2} = u(x_1, 1) = 0, \qquad \text{and} \qquad u(x) \text{ is outgoing at infinity,} \tag{2.4}$$

where the waveguide Ω is given by

$$\Omega = \{x = (x_1, x_2): 0 \leqslant x_1 < \infty,\ 0 \leqslant x_2 \leqslant 1\}. \tag{2.5}$$

Assuming that $n(x) \equiv 1$ for x_1 sufficiently large, say $x_1 \geqslant x^0$, the appropriate outgoing radiation condition is expressed by the fact that u may be represented as a sum of outgoing modal solutions for $x_1 \geqslant x^0$:

$$u(x) = \sum_{j=1}^{\infty} a_j e^{iK_j x_1} \cos\left(j - \frac{1}{2}\right) \pi x_2, \tag{2.6}$$

where

$$\begin{aligned} K_j &= i\sqrt{((j-\tfrac{1}{2})\pi)^2 - K^2} \qquad \text{for} \quad K < (j-\tfrac{1}{2})\pi \\ &= \sqrt{K^2 - ((j-\tfrac{1}{2})\pi)^2} \qquad \text{for} \quad K > (j-\tfrac{1}{2})\pi \end{aligned} \tag{2.7}$$

and $K \neq (j-\frac{1}{2})\pi$, $j = 1, 2, \ldots$.

Note that the modes corresponding to $(j-\frac{1}{2})\pi > K$ are exponentially decaying and hence may be neglected in the far field. The remaining modes are propagating. It can be seen that (2.4) may be replaced by a Helmholtz boundary value problem on the bounded domain $\Omega_{x^\infty} = \{x = (x_1, x_2): 0 \leqslant x_1 \leqslant x^\infty,\ 0 \leqslant x_2 \leqslant 1\}$, where $x^\infty > x^0$. In this case the outgoing radiation condition is replaced by a boundary condition on $x_1 = x^\infty$ of the form $\partial u/\partial x_1 = T(u)$, where T may be a global boundary operator ([8] or [10]) or a local boundary operator ([2] or [14]). We shall briefly discuss these boundary operators in the next section (see (3.13)).

Physical models exhibiting nearly one-way propagation generally satisfy the following two conditions: (i) most of the energy propagates in a narrow angle band about a fixed direction (not necessarily the horizontal), and (ii) backscattering is negligible. To illustrate the narrow angle condition (i), consider a model in which most of the x_1-propagation of the solution in a region $B \subset \Omega$ is at an angle of α radians with respect to the positive x_1 axis with $|\alpha| \ll \pi/2$. Hence given a fixed integer $m \geqslant 1$, we assume that the solution is given by

$$u(x) = e^{i(K\cos\alpha)x_1} v(x), \tag{2.8}$$

where the x_1-derivatives of v are sufficiently small in B that

$$\int_B |D_{x_1}^j D_{x_2}^{m-j} u(x)|^2\, dx \approx |K\cos\alpha|^{2j} \int_B |D_{x_2}^{m-j} v(x)|^2\, dx, \qquad j = 1, \ldots, m. \tag{2.9}$$

Set $U(x) = e^{-iKx_1}u(x) = e^{iK(\cos\alpha - 1)x_1}v(x)$. Since $|\cos\alpha - 1| \ll 1$, it follows readily from the assumptions on v that (2.3) holds with $\rho_1 = x_1$ and $K' = K$. Note that if the x_2-derivatives of v are sufficiently small, then (2.3′) holds as well.

Condition (i) is closely related to the modal expansion for u given by (2.6) and (2.7) for $x^0 \leqslant x_1 \leqslant x^\infty$. Suppose that the exponentially decaying modes are negligible for $x_1 = x$. Thus

$$u(x) = \sum_{j=1}^{M} a_j e^{iK_j x_1} \cos(j - \tfrac{1}{2})\pi x_2 \qquad \text{with} \quad (M - \tfrac{1}{2})\pi < K. \tag{2.10}$$

Suppose that K' is chosen so that $|K' - K_j| \ll K_j$ for $j = 1,\ldots, M$. Then (2.3) holds for $x^0 \leqslant x_1 \leqslant x^\infty$ with $U(x) = e^{-iK'x_1}u(x)$ and $\rho_1 = x_1$. For example, if all of the K_j present in (2.10) are clustered together, then K' may be defined as the average of the modal wave numbers

$$K' = \frac{1}{M}\sum_{j=1}^{M} K_j. \tag{2.11}$$

This is the case, e.g., if $M \ll K$ since each $K_j \approx K$. (In this case, (2.3′) also holds.)

Similar results hold if n is not constant but is only a function of x_2 for $x^0 \leqslant x_1 \leqslant x^\infty$. In this horizontally stratified case, the K_j might have to be calculated numerically as the solution of an eigenvalue problem. (It is not necessary, however, to calculate the eigenfunctions.) If $n(x_1, x_2)$ or other inhomogeneities vary slowly with x_1, it may be necessary to calculate the K_j corresponding to a few different values of x_1.

Remark 2.1. If more information is known about the coefficients a_j in (2.10), then better values of K' may be obtained than that given by (2.11). For example, K' may be defined as the weighted average of the modal wave numbers

$$K' = \frac{\sum_{j=1}^{M} |a_j| K_j}{\sum_{j=1}^{M} |a_j|}. \tag{2.12}$$

The selection of an optimal reference wave number, K', is treated in more detail in [16] in connection with the parabolic approximation. The heuristic argument for determining K' in [16] is based on Rayleigh's principle for progressive waves and appears to be applicable in the present context as well. For the solution u given by (2.10), the value of K' predicted in [16] is given by

$$K'^2 = \frac{\sum_{j=1}^{M} |a_j|^2 K_j^2}{\sum_{j=1}^{M} |a_j|^2}. \tag{2.13}$$

We shall see by means of numerical examples in Section 4 that while better choices of K' help in reducing the number of gridpoints, even the choice given by (2.11) is extremely effective.

We next consider condition (ii) above. Very little backscattering is present in B provided all lateral inhomogeneities in B (i.e., those in the x_1 direction) are slowly varying and any discontinuities are sufficiently small. This includes x_1-variations in $n(x)$ as well as all boundaries and interfaces contained in B. Consider, e.g., problem (2.4) with $n(x) \equiv 1$ for $x_1 \leqslant x^C < x^\infty$ and $n(x) \equiv 1+\varepsilon$ (with small $\varepsilon > 0$) for $x^C < x_1 \leqslant x^\infty$. We assume that $(M-\frac{1}{2})\pi < K < (M+\frac{1}{2})\pi$ (so that there are at most M propagating modes in the solution). We also assume that all exponentially decaying modes are negligible and the data g is such that only the first mode, $e^{iK_1 x_1}\cos(\pi x_2/2)$, is initially propagating outward.

It is easily seen that

$$u(x) = e^{iK_1 x_1}\cos\frac{\pi x_2}{2} + \sum_{j=1}^{M} R_j(\varepsilon)\, e^{-iK_j x_1}\cos\left(j-\frac{1}{2}\right)\pi x_2 \qquad \text{for } x_1 \leqslant x^C, \tag{2.14}$$

$$\sum_{j=1}^{M} T_j(\varepsilon)\, e^{iK_j x_1}\cos\sqrt{((j-\tfrac{1}{2})\pi)^2 + K^2\varepsilon} \qquad \text{for } x^C < x_1,$$

where $R_j(\varepsilon)$ and $T_j(\varepsilon)$ are the reflection and transmission coefficients, respectively, due to the jump in $n(x)$ at $x_1 = x^C$. Using (2.14) and the fact that u and $\partial u/\partial x_1$ are continuous at $x_1 = x^C$, it may be seen that $R_j(\varepsilon) = O(K^2\varepsilon)$ and $T_j(\varepsilon) = O(K^2\varepsilon)$ for $K^2\varepsilon$ small. Thus, condition (2.3) holds with $B = \Omega$, $\rho = x_1$, and $K' = K_1$ provided $K^2\varepsilon$ is sufficiently small. Observe that even if $K^2\varepsilon$ is not small, condition (2.3) can still hold with $B = \{x = (x_1, x_2): x^C < x_1 \leqslant x^\infty, 0 \leqslant x_2 \leqslant 1\}$ and K' suitably chosen since u will be a sum of outgoing modes in this region.

Remark 2.2. The condition of nearly one-way wave propagation holds for a wide variety of complicated problems in two and three dimensions. In fact, it can be easily seen that the applicability of typical outgoing radiation boundary conditions employed in practice (see [12]) is closely related to the existence of a principal propagation direction outside of some bounded set. This condition will often be valid to varying degrees in different subregions (i.e., $\delta = \delta(B)$ in (2.3)). This can be due to the varying effects of inhomogeneities and evanescent modes in different subregions. This can also follow when the outgoing waves are most conveniently expressed asymptotically using far field approximations. (For example, we saw this in connection with problem (1.1), where the outgoing solution in cylindrical coordinates was expressed in terms of a Hankel function.)

We close this section by considering the scattering of waves from a bounded obstacle in three dimensions. The model problem is given by

$$(-\Delta - K^2 n(x))\, u(x) = f(x) \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = g(x) \text{ on } \partial\Omega',$$
$$\frac{\partial u(x)}{\partial r} + (r^{-1} - iK)\, u(x) = o(r^{-1}) \text{ as } r = |x| \to \infty, \tag{2.15}$$

where $\Omega \subset R^3$ is the complement of a bounded obstacle Ω', $f \equiv 0$ and $n(x) \equiv 1$ outside of some bounded set, and we employ spherical polar coordinates (r, ω, θ) to represent an arbitrary point x. It follows from Green's theorem using the argument in [1] that

$$u(x) = e^{iKr} \sum_{j=1}^{\infty} \frac{a_j(\omega, \theta, K)}{r^j} \qquad \text{for } r = |x| \text{ sufficiently large.}$$

Hence $U(x) = e^{-iKr} u(x)$ satisfies the following estimate for each derivative D^α:

$$|D^\alpha U(x)| \leqslant \frac{C}{r^{|\alpha|+1}} \qquad \text{for } r \text{ sufficiently large.} \tag{2.16}$$

It may thus be readily seen that (2.3) holds with $\rho_1 = r$ for regions B sufficiently far away from the inhomogeneities. Therefore, the solution of the exterior problem (2.14) exibits nearly one-way propagation in the outgoing radial direction sufficiently far away from the obstacle.

3. The Numerical Methods

In this section we describe some finite element methods for efficiently treating nearly one-way propagation in the sense of definition (2.2), (2.3). We begin by very briefly describing the standard finite element method (SFEM) in connection with the boundary value problem

$$\begin{gathered}(-\Delta - K^2 n(x))\, u(x) = f(x) \text{ in } R^2, \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \\ \frac{\partial u}{\partial n} = T(u) \text{ on } \Gamma_\infty \text{ with } \partial\Omega = \Gamma \cup \Gamma_\infty,\end{gathered} \tag{3.1}$$

where $\partial/\partial n$ denotes the derivative in the outgoing normal direction, $f(x)$, $n(x)$, and $\partial\Omega$ are smooth, and the boundary operator T incorporates the appropriate outgoing radiation condition on Γ_∞. For simplicitiy, we consider Cartesian coordinates, $x = (x_1, x_2)$, in two dimensions. The finite element method is based on expressing (3.1) in the following weak or variational formulation, obtained using integration by parts,

$$\begin{gathered}a(u, v) = (f, v)\ \forall v \text{ in } H^E \equiv H^1(\Omega), \qquad \text{where} \\ a(u, v) \equiv \int_\Omega (\nabla u(x) \cdot \nabla \overline{v(x)} - K^2 n(x)\, u(x)\, \overline{v(x)})\, dx \\ - \oint_{\Gamma_\infty} T(u)(x)\, \overline{v(x)}\, ds_x,\end{gathered} \tag{3.2}$$

where $(\ ,\ )$ denotes the $L^2(\Omega)$ inner product.

In order to approximately solve (3.2), we approximate H^E by a family of finite dimensional subspaces $S^h \subset H^E$, $h \in (0, h_0)$. The spaces S^h are typically constructed by subdividing Ω into simple subsets called elements, such as triangles or quadrilaterals. Let $\Delta x_1 (\Delta x_2)$ denote the maximum $x_1 (x_2)$ grid sizes over all the elements, t^h, in Ω. (For example, if t^h consists of right triangles or rectangles, then Δx_1 and Δx_2 denote the largest lengths of the horizontal and vertical sides, respectively, over all the t^h.) The parameter, h, is defined as $\max(\Delta x_1, \Delta x_2)$. The spaces S^h generally consist of sufficiently smooth functions v^h such that v^h restricted to t^h is a polynomial of degree $m-1$ for some fixed integer $m \geqslant 2$. The following approximation property holds for the finite element spaces usually employed in practice:

> If $v \in H^m(\Omega)$, then $\exists$ a v^h in S^h and a constant C, independent of v, Δx_1, and Δx_2 such that
>
> $$\|v - v^h\|_{L^2(\Omega)} + \Delta x_1 \|D_{x_1}(v - v^h)\|_{L^2(\Omega)} + \Delta x_2 \|D_{x_2}(v - v^h)\|_{L^2(\Omega)} \leqslant C \left(\sum_{j=0}^{m} (\Delta x_1)^{2j} (\Delta x_2)^{2(m-j)} \int_B |D_{x_1}^j D_{x_2}^{m-j} v(x)|^2 \, dx \right)^{1/2}. \tag{3.3}$$

The finite element approximation, u^h, to the solution, u, of (3.2) (or 3.1)) is defined as the solution of

$$a(u^h, v^h) = (f, v^h) \qquad \forall v^h \text{ in } S^h. \tag{3.4}$$

For comprehensive treatments of the finite element method, see [6] or [20].

Now suppose that (2.2) and (2.3) hold with $B = \Omega$ and $\rho_1 = x_1$, so that $u(x) = e^{iK'x_1} U(x)$. In view of (2.3) and (3.3), we see that fewer x_1 grid points (i.e., larger grid sizes, Δx_1) are needed for approximating U than are required for a similar approximation of u with the same accuracy. We are thus motivated to approximately solve for U instead of u using the finite element method. We propose two methods for doing this. Method I consists of first replacing $u(x)$ by $U(x) = e^{-iK'x_1} u(x)$ in the variational formulation, (3.2), to obtain

$$A_1(U, v) \equiv a(u, v) = (f, v) \qquad \forall v \text{ in } H^E. \tag{3.5}$$

Using the bilinear form $A_1(\,,\,)$ defined in (3.5), we may now discretize (3.5) using the finite element "trial" spaces S^h defined as before. We thus wish to obtain a function U_1^h in S^h satisfying

$$A_1(U_1^h, V^h) = (f, V^h) \qquad \forall V^h \text{ in } S'^h, \tag{3.6}$$

where S'^h is a suitable "test" space. For example, S'^h could be taken to be S^h (the usual piecewise polynomial space). Alternatively, we can define S'^h as the space $\tilde{S}^h \equiv \{\tilde{v}^h \mid \tilde{v}^h = e^{iK'x_1} v^h, v^h \text{ in } S^h\}$. Note that the approximate solution, $u_1^h(x)$, of (3.2) is given by $e^{iK'x_1} U_1^h(x)$ in $\tilde{S}^h$.

Method II consists of first substituting $u(x) = e^{iK'x_1}U(x)$ directly into the boundary value problem (3.1). Hence we obtain a new boundary value problem for U. We next approximate U by applying the finite element method to this resulting boundary value problem. Specifically, we first obtain an equivalent variational problem as before by integrating by parts:

$$A_2(U, v) = (f, v) \qquad \forall v \text{ in } H^E. \tag{3.7}$$

We now discretize (3.7) using the finite element trial space S^h to obtain the following problem for U_2^h in S^h,

$$A_2(U_2^h, V^h) = (f, V^h) \qquad \forall V^h \text{ in } S'^h, \tag{3.8}$$

where again S'^h is a suitable test space and the approximate solution of (3.1) is given by $u_2^h(x) = e^{iK'x_1}U_2^h(x)$ in $\tilde{S}^h$. Note that Method I and Method II yield different discretizations in general, so that $u_1^h \neq u_2^h$. Furthermore, the choice of test space S'^h will have an important bearing on the efficiency of the numerical method.

To clarify these ideas, we consider problem (3.1) for the case in which Ω is the waveguide section given by

$$\Omega = \{(x_1, x_2): 0 \leqslant x_1 \leqslant x^\infty, 0 \leqslant x_2 \leqslant 1\} \tag{3.9}$$

and T is an appropriate radiation boundary operator satisfying

$$T(e^{iK'x_1}v^h)(x^\infty, x_2) = e^{iK'x^\infty}T(v^h)(x^\infty, x_2) \qquad \forall x_2 \quad \text{in} \quad [0, 1].$$

This condition holds for typical radiation boundary operators, T, including those given by (3.13) below. Applying Method I with $S'^h = S^h$, we see from (3.2), (3.5), and (3.6) that

$$\begin{aligned} A_1(U_1^h, V^h) &\equiv a(e^{iK'x_1}U_1^h, V^h) \\ &\equiv \int_\Omega e^{iK'x_1}\left(\nabla U_1^h(x)\cdot\overline{\nabla V^h(x)} + iK'U_1^h(x)\,\overline{\frac{\partial V^h(x)}{\partial x_1}}\right. \\ &\quad \left. - K^2 n(x)\, U_1^h(x)\,\overline{V^h(x)}\right) dx \\ &\quad - e^{iK'x^\infty}\int_0^1 T(U_1^h)(x^\infty, x_2)\,\overline{V^h(x^\infty, x_2)}\, dx_2 \\ &= (f, V^h) \qquad \forall V^h \text{ in } S^h. \end{aligned} \tag{3.10a}$$

Applying Method I with $S'^h = \tilde{S}^h$ and noting that each v^h in $\tilde{S}^h$ can be expressed as $v^h(x) = e^{iK'x_1}V^h(x)$ with V^h in S^h, we obtain

$$
\begin{aligned}
A_1(U_1^h, v^h) &\equiv a(e^{iK'x_1}U_1^h, e^{iK'x_1}V^h) \\
&\equiv \int_\Omega \left(\nabla U_1^h(x) \cdot \overline{\nabla V^h(x)} + iK' \left(U_1^h(x) \frac{\overline{\partial V^h(x)}}{\partial x_1} \right.\right. \\
&\quad \left.\left. - \frac{\partial U_1^h(x)}{\partial x_1} \overline{V^h(x)} \right) + (K'^2 - K^2 n(x))\, U_1^h(x)\, \overline{V^h(x)} \right) dx \\
&\quad - \int_0^1 T(U_1^h)(x^\infty, x_2)\, \overline{V^h(x^\infty, x_2)}\, dx_2 = (F, V^h) \qquad \forall V^h \text{ in } S^h,
\end{aligned}
\tag{3.10b}
$$

where $F(x) \equiv e^{-iK'x_1} f(x)$. It is easily seen that (3.10a) and (3.10b) yield different discretizations.

We next consider Method II and note that (3.1) is transformed into the following boundary value problem for U:

$$
e^{iK'x_1} \left(-\Delta - 2iK' \frac{\partial}{\partial x_1} + K'^2 - K^2 n(x) \right) U(x) = f(x) \text{ in } \Omega,
$$

$$
\frac{\partial U(x_1, 1)}{\partial x_2} = \frac{\partial U(x_1, 0)}{\partial x_2} = 0, \qquad \frac{\partial U(0, x_2)}{\partial x_1} + iK' U(0, x_2) = 0,
$$

$$
\frac{\partial U(x^\infty, x_2)}{\partial x_1} + iK' U(x^\infty, x^2) = T(U)(x^\infty, x_2).
$$

We now apply the finite element method with test space $S'^h = \tilde{S}^h$ to this boundary value problem. Since each v^h in $\tilde{S}^h$ can be expressed as $v^h = e^{iK'x_1} V^h(x)$ with V^h in S^h, we obtain

$$
\begin{aligned}
A_2(U_2^h, v^h) &= \int_\Omega \left(\nabla U_2^h(x) \cdot \overline{\nabla V^h(x)} - 2iK' \frac{\partial U_2^h}{\partial x_1}(x)\, \overline{V^h(x)} \right. \\
&\quad \left. + (K'^2 - K^2 n(x))\, U_2^h(x)\, \overline{V^h(x)} \right) dx \\
&\quad - \int_0^1 T(U_2^h)(x^\infty, x_2)\, \overline{V^h(x^\infty, x_2)}\, dx_2 \\
&\quad + iK' \int_0^1 (U_2^h(x^\infty, x_2)\, \overline{V^h(x^\infty, x_2)} - U_2^h(0, x_2)\, \overline{V^h(0, x_2)})\, dx_2 \\
&= (F, V^h) \qquad \forall V^h \text{ in } S^h,
\end{aligned}
\tag{3.11}
$$

where $F(x) = e^{-iK'x_1} f(x)$. Integrating by parts, we see that

$$
\begin{aligned}
&\int_0^1 \int_0^{x^\infty} U_1^h(x) \frac{\overline{\partial V^h(x)}}{\partial x_1}\, dx_1\, dx_2 \\
&\quad = \int_0^1 \left(- \int_0^{x^\infty} \frac{\partial U_1^h}{\partial x_1}(x)\, \overline{V^h(x)}\, dx_1 + U_1^h(x^\infty, x_2) \right. \\
&\qquad \left. \times \overline{V^h(x^\infty, x^2)} - U_1^h(0, x_2)\, \overline{V^h(0, x_2)} \right) dx_2.
\end{aligned}
\tag{3.12}
$$

Combining (3.10b)–(3.12), we conclude that (3.10b) is the same as (3.11). Hence, Method I and Method II yield the same discretization (and approximate solution) when $S'^h = \tilde{S}^h$.

It is readily seen that Method II yields a discretization different from (3.10a) or (3.10b) when $S'^h = S^h$. Also note that no oscillatory functions such as $e^{iK'x_1}$ appear in the integrals used in the formation of the stiffness matrix when $S'^h = \tilde{S}^h$, whereas such oscillatory functions do appear when $S'^h \neq \tilde{S}^h$. Since typical numerical quadrature formulas frequently employed in calculating the stiffness matrx have difficulty with such oscillatory functions, it appears that the use of $\tilde{S}^h$ as a test space offers certain computational advantages. We shall see below that there are also advantages in using $\tilde{S}^h$ from stability considerations.

We next consider convergence rates for Methods I and II (with test space $S'^h = \tilde{S}^h$) applied to (3.1) with Ω given by (3.9). We assume that $n(x) \equiv 1$ and $f(x) \equiv 0$ for $x_1 \geqslant x^0$ with $x^0 < x^\infty$. For our radiation boundary operator, T, we choose either the impedance boundary operator

$$T(u)(x^\infty, x_2) = iK_J u(x^\infty, x_2) \tag{3.13a}$$

or the global boundary operator

$$T(u)(x^\infty, x_2) = \sum_{j=1}^{M} \frac{iK_j}{2} \int_0^1 u(x^\infty, x_2') \cos\left(j - \frac{1}{2}\right) \pi x_2' \, dx_2' \cos\left(j - \frac{1}{2}\right) \pi x_2, \tag{3.13b}$$

where K_j is defined in (2.7) and M is chosen large enough that T includes all propagating modes and a sufficient number of evanescent modes. The operator in (3.13a) is appropriate when only the Jth propagating mode is significant [2]. In both cases, the error due to the approximate boundary operator, T, is of order $O(e^{-d(K)x^\infty})$, where $d(K) = \min |K - K_j|$, $j = 1, 2, \ldots$.

THEOREM 3.1. *Suppose that u satisfies* (3.1) *with T given by* (3.13a) *or* (3.13b) *(with M sufficiently large) and $U(x) = e^{-iK'x_1}u(x)$ with $|K'| \leqslant CK$, C independent of K. Suppose that $B \subset \Omega$, $h = \max(\Delta x_1, \Delta x_2)$,* (3.3) *holds, and K is uniformly bounded away from each K_j by d, $j = 1, 2, \ldots$. Then there exist positive constants α (depending on d), $C_0(K)$, and $C_1(K)$ such that if*

$$C_0(K)\, Kh \text{ is sufficiently small,} \tag{3.14}$$

then there is a unique solution U^h of (3.10b) *and*

$$\|U - U^h\|_{L^2(\Omega)} + \Delta x_1 \|D_{x_1}(U - U^h)\|_{L^2(\Omega)} + \Delta x_2 \|D_{x_2}(U - U^h)\|_{L^2(\Omega)} \tag{3.15}$$

$$\leqslant C_1(K)\left(\left(\sum_{j=0}^{m} (\Delta x_1)^{2j} (\Delta x_2)^{2(m-j)} \int_\Omega |D_{x_1}^j D_{x_2}^{m-j} U(x)|^2 \, dx\right)^{1/2} + e^{-\alpha x^\infty} \|f\|_{L^2(\Omega)}\right).$$

$C_0(K)$ and $C_1(K)$ are independent of Δx_1, Δx_2 and U, and grow at worst polynomially with K.

We shall not prove Theorem 3.1 here. If the boundary operator T is given by (3.13a), the proof is very similar to that given in [11] for the exterior problem. (More precise information concerning the behavior of $C_0(K)$ and $C_1(K)$ as K increases can be obtained using the arguments and results in [3] and [11].) For T given by (3.13b), the technical difficulties associated with the global boundary operator may be treated using the arguments in [10]. Note that (except for the exponentially small boundary error), the convergence rate in (3.15) is optimal in the sense of the approximation condition (3.3) provided the stability condition (3.14) holds. In fact, the stiffness matrix becomes ill-conditioned when Kh increases and the discrete approximation may not exist when Kh is too large. Such a stability condition is typical for finite element methods applied to indefinite problems.

In view of the stability condition, (3.14), we see that even if all of the x_1-dependence is factored out of $u(x)$ (so that U is only a function of x_2), it is not clear that we should obtain a reasonable approximation to U unless $K\Delta x_1$ is sufficiently small. Our numerical results indicate that, in fact, we do not obtain reasonable approximations for $K\Delta x_1$ large when the test space $S'^h \neq \tilde{S}^h$. However, as our numerical results in the next section demonstrate, this stability problem does not cause serious difficulties when $S'^h = \tilde{S}^h$. To get an idea of why this is true, as well as some insight into the proof of Theorem 3.1, we examine the simple one-dimensional problem

$$\begin{aligned} &-\frac{d^2u(x)}{dx^2} - K^2u(x) = 0 \text{ in } \Omega \equiv [0, 1], \\ &\frac{du(0)}{dx} = iKC_0, \qquad \text{and} \qquad \frac{du(1)}{dx} - iKu(1) = 0, \end{aligned} \tag{3.16}$$

where C_0 is real. The solution of (3.16) is given by $u(x) = C_0 e^{iKx}$. Now suppose we have a family, $S^h \subset H^1(\Omega)$, of standard piecewise polynomial finite element spaces of degree $m-1$ with $m \geqslant 2$.

THEOREM 3.2. *If we apply Method* I *or* II *to problem* (3.16) *with* $K' = K$ *and test space* $S'^h = \tilde{S}^h \equiv e^{iK'x}S^h$, *then there exists a unique approximate solution* $U^h(x)$ $\forall h$ *in* $(0, 1]$ *and*

$$\|U - U^h\|_{H^1(\Omega)} = 0. \tag{3.17}$$

We see from this theorem that no stability constraint such as (3.14) is necessary for (3.17) to hold. Theorem 3.2 will be proved in the Appendix. It can be seen from the proof that if $|K' - K|$ is small, we may choose $h = \Delta x$ to be large even for large K. It can also be seen that the proof fails if $S'^h \neq \tilde{S}^h$. In such cases, (3.17) holds only if K^2h is sufficiently small (see Remark A.1 in the Appendix).

Remark 3.1. Methods I and II can, of course, be readily extended to more complicated two- and three-dimensional problems. In view of the arguments in this and

the previous section and the numerical results in the next section, we expect that these methods should yield considerable improvements over standard discretization methods provided there is negligible backscattering. However, as indicated in Remark 2.2, the extent to which condition (2.3) holds (i.e., the size of δ) in different subregions of the computational domain depends on various physical aspects of the problem. Hence, the proper choice of mesh sizes in different subregions depends on these factors. A general and systematic approach for determining appropriate mesh sizes and other relevant parameters can be developed with the use of adaptive discretization methods. This is discussed in more detail in [12]. An adaptive discretization method has been developed and implemented, [17], for treating elliptic problems on unbounded regions.

As an example of the use of non-uniform mesh sizes, we consider the three-dimensional exterior scattering problem (2.15). We very briefly describe how this problem was treated in [11] using Method I. We first replace the unbounded domain Ω by a bounded domain Ω_R with outer boundary given by $\Gamma_R \equiv \{x: |x| = R\}$. The radiation condition is replaced by the approximate radiation boundary condition, $\partial u_R/\partial r = (iK - 1/R)\, u_R$ on Γ_R, where u_R denotes the solution of the new problem on Ω_R. It was proved in [11] that with $U_R \equiv e^{-iKr} u_R$ and B an arbitrary fixed subset of Ω_R, we have

$$\| U - U_R \|_{L^2(B)} = O(R^{-2}). \tag{3.18}$$

In view of (2.16) and (3.18), we expect that U_R can be efficiently approximated using Method I with successively larger radial mesh sizes in regions further away from the origin. This was proved rigorously in [11] by systematically partitioning Ω_R into annuli Ω^i with successively larger radial mesh sizes on each Ω^i, so that the resulting finite element space has optimal dimension $O(h^{-3})$, independent of R. It was shown that optimal convergence rates hold in this case. We refer to [11] for a detailed description and analysis of the method and to the next section for numerical results indicating the effectiveness of the method.

4. Numerical Results

In this section we shall demonstrate the effectiveness of the finite element methods developed in Section 3 to treat nearly one-way wave propagation. We first demonstrate typical results for propagation in the two-dimensional waveguide section, Ω, given by (3.9) with $x^\infty = 1$. The model problem is

$$\begin{aligned} &(-\Delta - K^2)\, u = 0 \text{ in } \Omega, \qquad \frac{\partial u(x_1, 0)}{\partial x_2} = u(x_1, 1) = 0, \\ &\frac{\partial u(0, x_2)}{\partial x_1} = g(x_2), \qquad \text{and} \qquad \frac{\partial u(1, x_2)}{\partial x_1} = T(u)(1, x_2), \end{aligned} \tag{4.1}$$

where the radiation boundary operator, T, is given by (3.13b). The datum, g, is chosen so that the exact solution is given by

$$u(x) = \sum_{j=1}^{M} a_j e^{iK_j x_1} \cos\left(j - \frac{1}{2}\right)\pi x_2 \tag{4.2}$$

with K_j given by (2.7).

We approximately solve (4.1) using the standard finite element method (SFEM) with trial and test space given by S^h, consisting of continuous piecewise linear functions defined on right triangles. We also employ Method I or Method II described in Section 3 with test space given by $\tilde{S}^h = e^{iK'x_1}S^h$, where K' is suitably chosen. As shown in Section 3, Methods I and II are equivalent in this case and our finite element solution approximates $U(x) \equiv e^{-iK'x_1}u(x)$, obtained by factoring out $e^{iK'x_1}$ from the exact solution. We refer to this method as FFEM. We employ a uniform $N_{x_1}xN_{x_2}$ grid (hence the grid sizes are $1/(N_{x_1}-1)$ and $1/(N_{x_2}-1)$) and we measure the normalized mean-square discretization error, E_2. The resulting linear equations are solved using the iterative method developed in [2].

For our first example, we choose data so that the exact solution is $u(x) = e^{iK_j x_1}\cos(j-\frac{1}{2})\pi x_2$ for some $j \geqslant 1$. We set $K' = K_j$, so that $U(x) \equiv e^{-iK'x_1}u(x)$ is independent of x_1 and we compare SFEM with Method I using test spaces $\tilde{S}^h$ (FFEM) and S^h. We see from Table I that for $j=4$, $K=24$, and $N_{x_2}=65$, the errors grow very slowly using FFEM as Δx_1 increases. Hence, we obtain reliable approximations with very few x_1 grid points. On the other hand we see from the last three entries that the errors increase dramatically when Δx_1 increases using Method I with test space S^h. We also see that the worst results are obtained using SFEM, as expected. Similar conclusions follow from Table II with $K=48$. The errors are slightly less in this case for FFEM since K is further away from the cutoff frequency, K_4, present in the solution. For the other two methods, the error increases with K. Finally, we see from Table III with $j=12$, $N_{x_2}=257$ and 512, and $K=40$ and 80 that FFEM

TABLE I

$K=24$, $N_{x_2}=65$, $j=4$

Method	N_{x_1}	E_2 (%)
SFEM	65	13.3
FFEM	17	1.0
FFEM	9	1.7
FFEM	3	4.1
Method I[a]	33	1.6
Method I[a]	17	5.0
Method I[a]	9	44.5

[a] The test space is S^h.

TABLE II

$K = 48,\ N_{x_2} = 65,\ j = 4$

Method	N_{x_1}	E_2 (%)
SFEM	65	69.0
FFEM	17	0.5
FFEM	9	1.2
FFEM	3	3.2
Method I[a]	65	1.4
Method I[a]	33	10.8
Method I[a]	17	>100

[a] The test space is S^h.

again yields reliable approximations with very few x_1 grid points, provided N_{x_2} is sufficiently large.

We see from Tables I and II that FFEM yields the best results. In fact, we have found this to be generally true in all of our test cases. In the remainder of this section we shall only compare SFEM and FFEM. For our next example, we consider the same solution, $u(x) = e^{iK_j x_1} \cos(j - \frac{1}{2})\pi x_2$, as in the first example. This time, however, we set $K' = K_j \cos\alpha$ with $0 < \alpha \leqslant \pi/2$ and $U(x) = e^{-iK' x_1} u(x)$. As α increases from 0 to $\pi/2$, we expect the improvement due to FFEM to diminish. In fact, for $\alpha = \pi/2$ the method reduces to SFEM.

Note. The standard paraxial approximation is valid provided $\alpha \leqslant \pi/12$ [15]. However, for $\alpha > \pi/12$ the usual parabolic equation may no longer be an adequate approximation to the Helmholtz equation. Recently, modifications of this method have been developed that give adequate approximations to the Helmholtz equation provided $\alpha < \pi/4$ [15]. It is clear that FFEM is more generally applicable, since this is a full angle approximation.

TABLE III

FFEM Is Used with $j = 12$

N_{x_2}	K	N_{x_1}	E_2 (%)
512	40	9	2.6
512	40	3	4.3
257	40	3	10.2
512	80	9	1.7
512	80	3	4.0
257	80	3	8.0

TABLE IV

$K = 12$, $N_{x_2} = 65$, $j = 4$

α (radians)	N_{x_1}	E_2 (%)
$\pi/2$ (SFEM)	33	5.3
$4\pi/9$	33	4.1
$\pi/3$	17	5.0
$\pi/4$	9	6.1
$\pi/12$	5	5.0

In Tables IV–VI, we demonstrate typical results for $j = 4$, $N_{x_2} = 65$, and various angles α and wave numbers K. Our first observation is that for $\alpha \leqslant \pi/12$, N_{x_1} can be chosen very small. In such cases, it might be more efficient to solve the resulting linear equations using a banded Gaussian elimination solver instead of an iterative method. Furthermore, even for angles α for which parabolic equation type methods fail, there is still a substantial reduction in the number of x_1 grid points.

We also observe that the improvement of FFEM over SFEM appears to increase with K (even for α close to $\pi/2$), although in general the error increases with K for a fixed mesh size. (This is also the case if the domain size increases.) It was shown in [3] for SFEM that when K is sufficiently large and uniformly bounded away from all cutoff frequencies, the error E_2 behaves like $O(K^3h^2)$ for a fixed domain. Hence, if K is doubled and both N_{x_1} and N_{x_2} are halved, the error is doubled. We see from Tables IV–VI (where N_{x_2} is kept fixed) that the error grows more slowly with increasing K using FFEM than using SFEM. On the other hand, as $K \to \infty$, the rate of increase of the error using FFEM approaches that predicted in [3] for SFEM.

For our third set of numerical experiments we consider a solution of the form (4.2) with $K > (M - \frac{1}{2})\pi$. This is equivalent to assuming that all attenuating modes and all other modes corresponding to $j > M$ are negligible. In Table VII, we com-

TABLE V

$K = 24$, $N_{x_2} = 65$, $j = 4$

α (radians)	N_{x_1}	E_2 (%)
$\pi/2$	65	13.3
$4\pi/9$	65	8.2
$\pi/3$	33	6.6
$\pi/4$	17	5.5
$\pi/12$	5	5.5

TABLE VI

$N_{x_2} = 65$, $j = 4$

K	α (radians)	N_{x_1}	E_2 (%)
48	$\pi/3$	65	8.8
48	$\pi/4$	33	5.4
96	$\pi/4$	65	8.9
48	$\pi/12$	5	2.3
96	$\pi/12$	5	11.7

pute the error E_2 for $a_j = 1$, $j = 1,..., M$, and various values of M, N_{x_1}, N_{x_2}, and K.

$$K_1' = \frac{1}{M} \sum_{j=1}^{M} K_j. \tag{4.3}$$

(In all cases, we have found that the errors using FFEM with $K' = K_1'$ are substantially less than those obtained using SFEM.) We see from Table VII that for fixed M, N_{x_1}, and N_{x_2}, the errors decrease as K increases. This is contrary to results obtained using standard discretization methods and is due to the fact that $K_j \to K$, $j = 1,..., M$, and $K_1' \to K$ as $K \to \infty$. Hence for $K \gg M$, $U(x) = e^{-iK_1' x_1} u(x)$ is nearly independent of x_1 and very few x_1 grid points are needed. On the other hand, for K fixed the error increases as M increases.

In general, we may not know which propagating modes are important in the solution. In such cases we could simply define K' by (4.3) with M denoting the

TABLE VII

FFEM with $K' = K_1'$ (4.3)

M	N_{x_1}	N_{x_2}	K	E_2 (%)
4	17	65	15	5.1
4	17	65	24	1.9
4	9	65	24	2.9
8	17	65	30	26.5
8	17	65	45	7.0
8	17	65	100	1.8
8	9	65	100	3.9
8	5	65	100	11.2
12	17	129	45	47.0
12	17	129	100	4.0
12	17	129	150	2.0

TABLE VIII

SFEM and FFEM with K' Given by K'_1 (4.3), K'_2 (4.4), and K'_3 (4.5); $N_{x_2} = 65$

M	K	Method	N_{x_1}	E_2 (%)
8	26	SFEM	65	11.3
8	26	FFEM (K'_1)	9	10.5
8	26	FFEM (K'_2)	9	3.8
8	26	FFEM (K'_3)	9	3.4
8	26	FFEM (K'_2)	5	10.9
8	26	FFEM (K'_3)	5	6.8
12	38	SFEM	65	31.8
12	38	FFEM (K'_1)	9	11.7
12	38	FFEM (K'_2)	9	3.3
12	38	FFEM (K'_3)	9	3.0
12	38	FFEM (K'_2)	5	10.8
12	38	FFEM (K'_3)	5	5.5

number of possible propagating modes. We would still expect FFEM to yield a substantial improvement over SFEM. On the other hand, better choices of K' could be made if we have more information about the solution. We demonstrate this in Tables VIII and IX. In Table VIII we compute E_2 for the smooth solution,

$$u(x) = \sum_{j=1}^{M} \frac{1}{j^2} e^{iK_j x_1} \cos\left(j - \frac{1}{2}\right) \pi x_2,$$

TABLE IX

SFEM and FFEM with K' Given by K'_1 (4.3), K'_2 (4.4), K'_3 (4.5), and K'_4 (4.6); $N_{x_2} = 129$

M	K	Method	N_{x_1}	E_2 (%)
8	30	SFEM	33	70.0
8	30	FFEM (K'_1)	33	8.5
8	30	FFEM (K'_2)	33	7.4
8	30	FFEM (K'_3)	33	6.8
8	30	FFEM (K'_4)	33	5.0
12	45	SFEM	33	>100
12	45	FFEM (K'_1)	33	29.4
12	45	FFEM (K'_2)	33	24.2
12	45	FFEM (K'_3)	33	19.6
12	45	FFEM (K'_4)	33	8.8

using SFEM and FFEM with $K' = K_1'$ defined by (4.3) and $M = 8$ and 12. In view of (2.12) and (2.13), we also define the weighted average

$$K_2' = \frac{\sum_{j=1}^{M} |a_j| K_j}{\sum_{j=1}^{M} |a_j|} \tag{4.4}$$

and the value derived in [16]

$$K_3' = \sqrt{\frac{\sum_{j=1}^{M} |a_j|^2 K_j^2}{\sum_{j=1}^{M} |a_j|^2}} \tag{4.5}$$

We again apply FFEM. It is clear from Table VIII that FFEM using either K_1', K_2' or K_3' is superior to SFEM and K_3' gives the best results.

In Table IX we consider the solution

$$u(x) = \sum_{j=1}^{M} \frac{e^{iK_j x_1} \cos\left(j - \frac{1}{2}\right) \pi x_2}{\sqrt{K^2 - \left(\left(j - \frac{1}{2}\right) \pi\right)^2}}$$

with $M = 8$ and 12. In this case (with $(M - \frac{1}{2})\,\pi < K < (M + \frac{1}{2})\,\pi$), $u(x)$ corresponds to the propagating part of the Greens function $G(x; x')$ for problem (2.4) in the semi-infinite waveguide (2.5), with source point $x' = (0, 0)$. This solution behaves differently than the solutions considered before since the high propagating modes now contain most of the energy. Hence, we define the average of the four highest modal wave numbers

$$K_4' = \frac{1}{4} \sum_{j=M-3}^{M} K_j \tag{4.6}$$

and compare FFEM using K_1', K_2', K_3', and K_4'. Again, we see from Table VIII that K_1', K_2', and K_3' yield considerably better results than SFEM. However, this time the best results are obtained using K_4'. These results indicate that the choice of the most suitable K', as well as the appropriate number of grid points, depends strongly on the oscillatory nature of the solution.

For our final example, we illustrate the use of non-uniform mesh sizes combined with FFEM in connection with the exterior scattering problem, (2.15). Consider the following axially symmetric three-dimensional model problem. Let Ω denote the region exterior to the sphere, $\Gamma \equiv \{x: |x| = r_0\}$, and let Ω^c denote the interior of this sphere. The boundary value problem is given by

$$(-\Delta - K^2)\, u = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = g \text{ on } \Gamma, \qquad \text{and } u \text{ is outgoing at infinity.} \tag{4.7}$$

The data, g, corresponds to the solution

$$u(x) = \frac{e^{iK|x - x_s|}}{4\pi\, |x - x_s|}, \tag{4.8}$$

where x_s is a fixed axial point inside Ω^c. We replace Ω by a bounded domain by introducing the artificial outer boundary, $\Gamma_R \equiv \{x: |x| = R\}$. Employing spherical polar coordinates and axial symmetry, we thus consider the rectangular computational domain, $\Omega_R \equiv \{(r, \theta): r_0 \leqslant r \leqslant R,\ 0 \leqslant \theta \leqslant \pi\}$.

We approximate the outgoing Sommerfeld radiation condition by either of the following two boundary conditions on Γ_R:

$$B_1 u \equiv \frac{\partial u}{\partial r} + \left(\frac{1}{r} - iK\right) u = 0 \qquad \text{on} \quad \Gamma_R \tag{4.9a}$$

or

$$B_2 u \equiv \frac{\partial^2 u}{\partial r^2} + \left(\frac{4}{r} - 2iK\right) \frac{\partial u}{\partial r} + \left(\frac{3}{r} - 4iK\right) \frac{u}{r} - K^2 u = 0 \qquad \text{on } \Gamma_R. \tag{4.9b}$$

The boundary operators, B_1 and B_2, correspond to the first two in a hierarchy of approximate boundary operators developed in [4]. Each operator, B_m, introduces a boundary error that decays like $R^{-\gamma_m}$ with $\gamma_m > 0$ and γ_m increasing with m (see, e.g., (3.18) for $m = 1$). For details concerning these boundary operators and their implementation in a variational formulation, we refer to [4]. It thus follows that the two main sources of error in the numerical solution of (4.7) are the boundary error and the discretization error. For smaller wave numbers the boundary error usually dominates, whereas for larger wave numbers the discretization error dominates.

We again approximately solve the resulting boundary value problem using SFEM and FFEM. For SFEM, we employ continuous piecewise linear trial and test functions defined on a triangular partition in (r, θ) coordinates. When using FFEM, we first factor out the far field radial dependence e^{iKr}. As described at the end of Section 3, in order to effectively apply FFEM we must grade the mesh using successively larger mesh sizes further away from the origin. A method for doing this was developed and analyzed in [11]. For our sample computations we choose 6 radial grid points, set $R = 1.125$, $r_0 = 0.5$, and again measure E_2, the normalized mean-square error. The number of θ grid points is denoted by NTH and these points are equally spaced. When employing SFEM, we use the radial grid points $r_j = r_0 + ((R - r_0)/5)j$, $j = 0,..., 5$. When FFEM is employed, we grade the mesh in accordance with

$$r_1 = r_0 + h_0 \qquad \text{and} \qquad r_j = r_{j-1} + C_0 h_0 r_{j-1}^{3/2}, \qquad j = 2,..., 5, \tag{4.10}$$

where $h_0 = 0.047$ and $C_0 = 5$. We shall discuss the mesh grading further in Remark 4.1 below.

Typical numerical results are presented in Table X for different values of K and x_s. For $K = 0$ the boundary error dominates and we employ the boundary operator B_2 in order to compare discretization errors. For all other values of K, it suffices to

TABLE X

SFEM and FFEM (with a Graded Mesh) for the Exterior Problem

			E_2 (%)	
K	NTH	x_s	SFEM	FFEM
0	41	0.2	0.6	0.6
12	81	0.2	32.4	2.2
24	161	0.2	>100	2.5
48	161	0.2	>100	2.2
96	161	0.2	>100	4.7
120	161	0.2	>100	8.4
0	41	0.3	1.1	0.9
12	81	0.3	31.9	4.1
24	161	0.3	>100	8.5
48	161	0.3	>100	21.6
0	41	0.4	2.7	1.6
12	81	0.4	31.0	8.3
24	161	0.4	>100	26.8

use B_1. It is clear from (4.8) that for small values of x_s, the r-dependence of $U(x) = e^{-iKr}u(x)$ is weak and hence very few radial grid points should suffice when using FFEM even for large K. This is confirmed in Table X. We also see that for all values of x_s and a wide range of wave numbers, the mesh grading combined with FFEM results in a dramatic reduction in the error for a fixed number of grid points. When K becomes too large, it is necessary to either insert more radial grid points or make R smaller and employ a higher order (more complicated) boundary operator. The use of higher order boundary operators does not diminish the improvements obtained by combining FFEM with an appropriate mesh grading.

Remark 4.1. To get an idea of the reduction in computations resulting from using FFEM instead of SFEM, note that most of the computational cost involves the solution of the resulting system of linear equations. Since this system of equations can be quite large, we have employed a preconditioned conjugate gradient iterative method (applied to the normal equations), developed in [2], for both SFEM and FFEM. We have seen that the number of grid points required to achieve comparable accuracy is considerably reduced using FFEM, say by a factor of $\lambda < 1$. Hence the storage requirements and operation count/iteration are reduced by approximately a factor of λ. It is also the case that the number of iterations is generally reduced using FFEM since the condition number of the resulting matrix is smaller. Furthermore, when the application of FFEM allows the use of a small number of grid points in one coordinate direction (as we have seen is frequently the case), even greater savings in cost can be obtained using a banded Gaussian elimination solver. Therefore, the reduction in total computational cost resulting

from using FFEM will typically be a factor $\lambda' < \lambda$. The value of λ' depends on various factors, including the choice of reference wave number and preconditioning.

Remark 4.2. The mesh grading defined by (4.10) involved constants, C_0 and h_0, that were chosen somewhat arbitrarily. (The exponent, $\frac{3}{2}$, was motivated by the theory in [11] since we are using piecewise linear finite elements.) Furthermore, note that the mesh grading chosen is independent of K and x_s, whereas we expect these parameters to be important considerations in the selection of the mesh size. A more systematic approach based on adaptive discretization methods could be quite useful in determining optimal grid sizes as well as other relevant parameters such as R. This was done in [17] using adaptive finite element methods in connection with the exterior problem for a class of positive definite elliptic operators. In the present context, this corresponds to $K=0$. In [17], suitably constructed error indicators and estimators were employed to determine optimal grid points as well as the size of the truncated domain. Furthermore, a preconditioned conjugate gradient method was employed at several steps of the process to iteratively improve the accuracy of

waveguides and exterior problems. We also intend to apply our methods to more complicated propagation models, including those with backscattering.

Appendix

Proof of Theorem 3.2. Setting $U(x)=e^{-iK'x}u(x)$, we obtain the following variational problem for U:

$$A(U, V) \equiv \int_0^1 \left(\frac{dU(x)}{dx}\overline{\frac{dV(x)}{dx}} + iK' \left(U(x)\overline{\frac{dV(x)}{dx}} - \frac{dU(x)}{dx}\overline{V(x)} \right) \right) dx$$

$$+ (K'^2 - K^2) \int_0^1 U(x)\,\overline{V(x)}\,dx - iKU(1)\,\overline{V(1)}$$

$$= -iKC_0\overline{V(0)} \qquad \forall V(x) = e^{-iK'x}v(x) \text{ in } H^1(\Omega). \tag{A.1}$$

We now apply Method I with test space $S'^h = \tilde{S}^h \equiv e^{iK'x}S^h$. Hence, letting U^h in S^h denote our approximation to U, we obtain the following variational problem for U^h,

$$A(U^h, V^h) = -iKC_0\overline{V^h(0)} \qquad \forall V^h(x) = e^{-iK'x}v^h(x) \text{ in } S^h, \tag{A.2}$$

where $A(\,,\,)$ is defined by (A.1). (As before, we can see that Method II with $S'^h = \tilde{S}^h$ yields the same discrete problem.)

Assume, for now, that there exists a solution U^h of (A.2) and set

$$E^h(x) = U(x) - U^h(x). \tag{A.3}$$

It follows from the definition of $A(\ ,\ )$ in (A.1) that

$$\int_0^1 \left|\frac{dE^h(x)}{dx}\right|^2 dx = A(E^h, E^h) + iK\,|E^h(1)|^2 - \int_0^1 \left(iK' \left(E^h(x)\,\overline{\frac{dE^h(x)}{dx}} - \frac{dE^h(x)}{dx}\,\overline{E^h(x)} \right) + (K'^2 - K^2)\,|E^h(x)|^2 \right) dx. \tag{A.4}$$

Now suppose that $K' = K$, so that

$$U(x) = C_0 \qquad \text{on} \quad \Omega. \tag{A.5}$$

Since U is real, we may assume that the finite element space S^h (and therefore U^h) is also real. Equating the real parts of (A.4) and applying (A.1) and (A.2), we deduce

$$\int_0^1 \left|\frac{dE^h(x)}{dx}\right|^2 dx = \mathrm{Re}(A(E^h, E^h)) = \mathrm{Re}(A(E^h, U - U^A)) \qquad \forall U^A \text{ in } S^h, \tag{A.6}$$

where Re( ) denotes the real part.

Since an approximation condition analogous to (3.3) holds for the finite element space S^h, we may apply the Schwarz inequality and the Sobolev inequality to the right side of (A.6) and choose a suitable U^A in S^h to conclude that

$$\begin{aligned} \int_0^1 \left|\frac{dE^h(x)}{dx}\right|^2 dx &\leqslant C(K)\,\|E^h\|_{H^1(\Omega)}\,\|U - U^A\|_{H^1(\Omega)} \\ &\leqslant C(K)\,h^{m-1}\,\|E^h\|_{H^1(\Omega)}\,|U|_{H^m(\Omega)} = 0, \end{aligned} \tag{A.7}$$

since U is constant and $|\ |_{H^m}$ only involves the mth order derivative of U. Combining (A.3) and (A.7), we see that $U^h(x)$ is a real contant, C', on Ω. Hence

$$u^h(x) \equiv e^{iKx} U^h(x) = C' e^{iKx} \tag{A.8}$$

is our approximate solution of (3.16). Using (A.2), it follows readily that

$$\begin{aligned} -iKC_0\,\overline{v^h(0)} = \int_\Omega \left(\frac{du^h(x)}{dx}\,\overline{\frac{dv^h(x)}{dx}} - K^2 u^h(x)\,\overline{v^h(x)} \right) dx \\ - iKu^h(1)\,\overline{v^h(1)} \qquad \forall v^h \text{ in } \tilde{S}^h. \end{aligned}$$

We next employ (A.8) and integration by parts and choose a v^h in $\tilde{S}^h$ such that $v^h(0) = 1$ and v^h vanishes near $x = 1$ to obtain

$$-iKC_0 = \int_0^1 \left(-\frac{d^2u^h(x)}{dx^2} - K^2 u^h(x) \right) \overline{v^h(x)}\,dx - \frac{du^h(x)}{dx}\,\overline{v^h(x)}\,|_{x=0} = -iKC'.$$

Hence $C' = C_0$ and

$$E^h(x) = 0 \qquad \text{on } \Omega. \tag{A.9}$$

It thus follows that (3.17) holds without assuming any stability condition such as (3.14). To see that there exists a U^h in S^h satisfying (A.2), it suffices to prove uniqueness since S^h is finite dimensional. If U^h satisfies (A.2) with $C_0 = 0$, then we see from (A.5) that $U(x) = 0$ on Ω. It now follows from (A.9) that $U^h(x) = 0$ on Ω and (A.2) is well-posed. Q.E.D.

Remark A.1. If Method I is employed with test space S^h instead of $\tilde{S}^h$, then the bilinear form $A(\ ,\ )$ defined in (A.1) is replaced by

$$A_1(U, V) \equiv \int_0^1 \left(\frac{dU(x)}{dx} \overline{\frac{dV(x)}{dx}} + iK' V(x) \overline{\frac{dV(x)}{dx}} - K^2 U(x) \overline{V(x)} \right) dx - iKU(1) \overline{V(1)}.$$

Using this bilinear form, we then obtain the following equation instead of (A.4):

$$\int_0^1 \left| \frac{dE^h(x)}{dx} \right|^2 dx = A_1(E^h, E^h) + iK |E^h(1)|^2 - \int_0^1 iK' E^h(x) \overline{\frac{dE^h(x)}{dx}}\, dx + K^2 \int_0^1 |E^h(x)|^2\, dx. \tag{A.10}$$

A standard duality argument from finite element theory (see [6]) can be applied to the last term in (A.10), resulting in the estimate

$$K^2 \int_0^1 |E^h(x)|^2\, dx \leqslant C(K)\, K^2 h^2 \, \|E^h\|^2_{H^1(\Omega)}. \tag{A.11}$$

Combining (A.10) and (A.11) with the proof of Theorem 3.2, we can show that (3.17) holds provided the stability condition (3.14) holds.

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